Electromagnetism: Electromagnetic Induction

FIZIKA SPhO Training

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Contents

1 Notes

In your EM lessons so far, we've only dealt with static situations (i.e. where $\mathbf{E}$ and $\mathbf{B}$ do not change with time). Now, we seek to understand the interesting physics behind what happens when we introduce time varying fields.

1.1 Induced Emf

We briefly talked about emf, ε, in DC circuits. But what exactly is emf?

1.1.1 What is Emf?

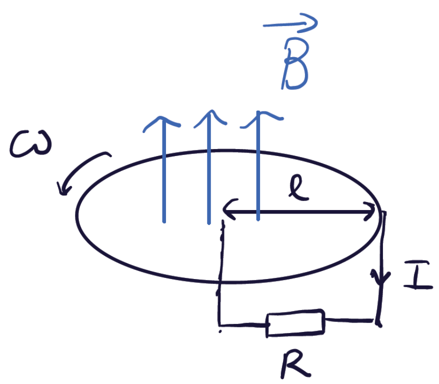
The emf is formally defined as

$$\varepsilon = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} \tag{1}$$

where $\mathbf{f}$ is the force per unit charge. In this definition, $\mathbf{f}$ can be any type of force of arbitrary origin (magnetic, electric etc).

Equation (1) alone already allows you to solve many problems! The example below illustrates how to apply it.

Example 1.1. A metal disk of radius l is parallel to the $x-y$ plane. It rotates about its central axis with a constant angular velocity ω. There is a constant magnetic field $\mathbf{B} = B\,\hat{\mathbf{z}}$ going through the disk. The centre of the disk and the edge of the disk are connected by a resistor of resistance R. Find the current flowing through the resistor.



To solve this, we just need to remember that

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \quad \Rightarrow \quad \mathbf{f} = \frac{\mathbf{F}}{q} = \mathbf{v} \times \mathbf{B} \quad \Rightarrow \quad |\mathbf{f}| = vB$$

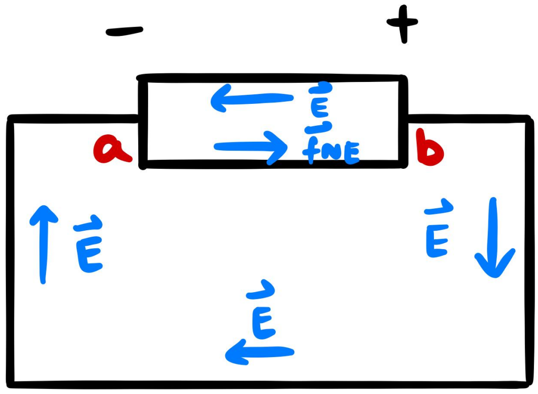
where the last equality follows as the velocity of the electrons in the plate, $\mathbf{v}$, is perpendicular to the magnetic field $\mathbf{B}$.

Thus, applying Equation (1),

$$\varepsilon = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \int_{\text{edge}}^{\text{center}} \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \int_0^l vB\,\mathrm{d}l = \int_0^l \omega Bl\,\mathrm{d}l = \frac{\omega Bl^2}{2} \quad \Rightarrow \quad I = \frac{\varepsilon}{R} = \frac{\omega Bl^2}{2R}$$

Hopefully everything made sense. However, you may be confused; why $\varepsilon = IR$ and not $V = IR$?

To resolve this, let's look at a battery and how electric fields work in it.



All contributions of $\mathbf{f}$ are either electrostatic or non-electrostatic:

$$\varepsilon = \oint \mathbf{f} \cdot \mathrm{d}\mathbf{l} = \oint \left(\mathbf{f}_{\text{electrostatic}} + \mathbf{f}_{\text{non-electrostatic}}\right) \cdot \mathrm{d}\mathbf{l} \tag{2}$$

Recall that our usual electrostatic $\mathbf{E}$ field is conservative. (Remember that the electric potential exists, which is why it is conservative!) This mathematically mandates that

$$\oint \mathbf{f}_{\text{electrostatic}} \cdot \mathrm{d}\mathbf{l} = 0 \tag{3}$$

Hence, defining our points a and b at the two terminals of the battery (as per the diagram above), Equation (2) simplifies into:

$$\varepsilon = \oint \mathbf{f}_{\text{non-electrostatic}} \cdot \mathrm{d}\mathbf{l} = \int_a^b \mathbf{f}_{\text{non-electrostatic}} \cdot \mathrm{d}\mathbf{l} \tag{4}$$

However, recall the definition of the electric potential:

$$V(b) - V(a) = -\int_b^a \mathbf{E} \cdot \mathrm{d}\mathbf{l} = \int_a^b \mathbf{E} \cdot \mathrm{d}l = \int_a^b \mathbf{f}_{\text{non-electrostatic}} \cdot \mathrm{d}l \tag{5}$$

Comparing Equations (4) and (5), we see that $V = \varepsilon$, hence the potential difference and emf are basically the same!

1.1.2 Faraday's and Lenz's Law

Faraday found that

$$\varepsilon = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} \tag{6}$$

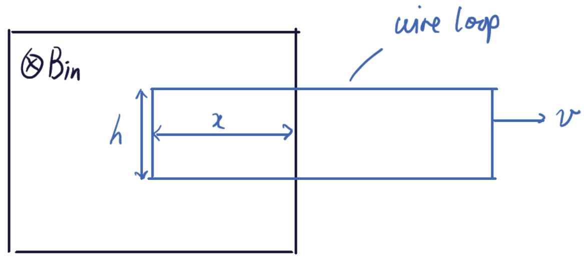
Equation (6) is known as **Faraday's Law.** This means that a changing magnetic flux through a loop will cause an induced emf in it.

Another important consequence is **Lenz's Law.** It states that the direction of the induced current will be such that it **opposes** the change that caused it. This is essentially COE, and accounts for the negative sign in Equation (6).

Remark. Equation (6) is very versatile! The loop can even move around or expand in size, and it still applies.

The example below illustrates how to apply it.

Example 1.2. A rectangular metal loop with a very long length and a width h is partially placed in a constant, uniform magnetic field B pointing into the page. A resistor R is part of this loop. Let the distance x be indicated as per the figure below. (i) Verify that both Equations (1) and (6) give the same expression for the induced emf. (ii) What is the direction of the induced current?



(i) Using Equation (1), we can perform a similar simplification as Example 1.1 to obtain

$$\varepsilon = \int vB\,\mathrm{d}l = Bhv$$

Using Equation (6), noting that B is constant,

$$\varepsilon = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = -B\frac{\mathrm{d}A}{\mathrm{d}t} = -Bh\frac{\mathrm{d}x}{\mathrm{d}t} = Bhv$$

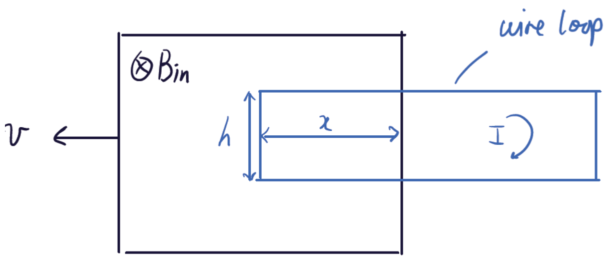
Hence, the results for ε given by both Equations (1) and (6) agree, as expected.

(ii) The direction of the induced current should oppose the change that caused it. As the magnetic flux into the page is decreasing, the induced current should increase the magnetic flux into the page. By the right hand grip rule, it will hence be **clockwise,** as seen from above.

1.1.3 Induced Electric Fields

Let's now consider a seemingly paradoxical situation.

Example 1.3. Consider moving a magnet into a circuit as shown below. As the magnet enters the circuit, $\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t}$ is non-zero, and hence ε is non-zero. But, what is the $\mathbf{f}$ in this case?



The resolution to this is that $\mathbf{f}$ comes from the **induced electric field!** That is, changes in flux will induce electric fields.

To be more exact, this is because **Faraday's Law** is written as

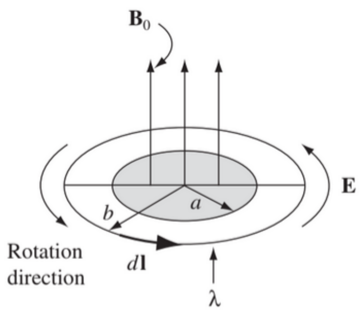
$$\oint \mathbf{E}\cdot\mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} \tag{7}$$

Hence, comparing this with Equation (1) and Equation (6), you can see that $\mathbf{f}$ in this case is just $\mathbf{E}$ (the induced electric field)!

Equation (7) appears very strange. It appears that this directly contradicts Equation (3), since the RHS is non-zero. This is because *induced* electric fields are actually **non-conservative!** (We cannot assign a potential function to them.)

Now that you know about the induced electric field, let's see how to apply this field to more complex situations.

Example 1.4 (Griffiths)**.** A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, as shown in the figure below, so that it is free to rotate (the spokes are made of some nonconducting material). In the central region, out to radius a, there is a uniform magnetic field B_0 pointing upwards. Now, someone turns this field off. What happens to the loop?



Let's first qualitatively reason what is happening. The changing magnetic field will induce a tangential electric field, as per Equation (7). This induced electric field exerts a tangential force on the charges along the rim, which causes a torque about the wheel's central axis, and the wheel starts to rotate.

According to Lenz's Law, the wheel rotates in a direction such that the field generated restores the upward flux. Hence, it will rotate in the anti-clockwise direction, as seen from above.

Applying Equation (7) to the loop at radius b,

$$\oint \mathbf{E} \cdot \mathrm{d}\mathbf{l} = E\,(2\pi b) = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = -\pi a^2 \frac{\mathrm{d}B}{\mathrm{d}t} \quad \Rightarrow \quad E = -\frac{a^2}{2b}\frac{\mathrm{d}B}{\mathrm{d}t}$$

The torque on a segment of length $\mathrm{d}l$ is hence

$$\mathrm{d}\tau = b\lambda E\,\mathrm{d}l$$

Integrating over the whole loop, the total torque is

$$\tau = b\lambda\left(-\frac{a^2}{2b}\frac{\mathrm{d}B}{\mathrm{d}t}\right)\left(\oint \mathrm{d}l\right) = b\lambda\left(-\frac{a^2}{2b}\frac{\mathrm{d}B}{\mathrm{d}t}\right)(2\pi b) = -b\lambda\pi a^2\frac{\mathrm{d}B}{\mathrm{d}t}$$

Hence, the angular momentum imparted to the wheel is

$$L = \int \tau\,\mathrm{d}t = -b\lambda\pi a^2 \int_{B_0}^{0} \mathrm{d}B = \lambda\pi a^2 b B_0$$

Interestingly, the angular momentum does not depend on the time you take to turn off the field. This means that the final angular velocity of the wheel is the same, regardless!

1.2 Inductance

We know that changing the current in a loop changes the magnetic field passing through it, and hence the magnetic flux. This causes an emf in the loop itself. We can quantify how this works using **inductance,** which is measured in henries (H).

1.2.1 Self-Inductance

Consider the Biot-Savart Law:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint \frac{\mathrm{d}\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3} \tag{8}$$

Let's think about how quantities scale with respect to each other. Clearly, from Equation (8), $B \sim I$, hence $\Phi_\mathrm{B} \sim B \sim I$. As $\varepsilon = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t}$, hence $\varepsilon \sim \frac{\mathrm{d}I}{\mathrm{d}t}$.

This suggests we may define a quantity L, called the **inductance,** such that

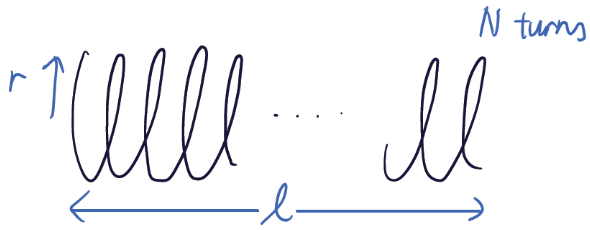
$$\varepsilon = -L\frac{\mathrm{d}I}{\mathrm{d}t} \tag{9}$$

To calculate inductance, a very convenient formula is often used:

$$L = \frac{\Phi_\mathrm{B}}{I} \tag{10}$$

Remark. Inductance is a **purely geometric property** and hence does not depend on current! It should only depend on quantities with length dimensions and μ_0. Also, when not specified, inductance refers to self-inductance.

Example 1.5. Find the self-inductance of a long solenoid of N turns, radius r, length l and current I.



This is a direct application of Equation (10):

$$\Phi_\mathrm{B} = NB\pi r^2 = N\pi r^2\left(\frac{\mu_0 NI}{l}\right) = \frac{\mu_0 N^2 \pi r^2 I}{l} \quad \Rightarrow \quad L = \frac{\Phi_\mathrm{B}}{I} = \frac{\mu_0 N^2 \pi r^2}{l}$$

As expected, apart from μ_0, the expression for L only depends on geometric properties (N, r, l).

1.2.2 Energy in Inductors

To oppose the back emf generated by an inductor, work must be done. This means that inductors store energy!

We can derive the **energy stored in an inductor** by considering the power:

$$P_\mathrm{L} = (-\varepsilon)\, I = L\frac{\mathrm{d}I}{\mathrm{d}t}I \quad \Rightarrow \quad U_\mathrm{L} = \int P_\mathrm{L}\,\mathrm{d}t = \int LI\frac{\mathrm{d}I}{\mathrm{d}t}\,\mathrm{d}t = \int LI\,\mathrm{d}I = \frac{1}{2}LI^2 \tag{11}$$

We can generalise this to volumes of space by introducing the **magnetic energy density,** u_B:

$$u_\text{B} = \frac{B^2}{2\mu_0} \tag{12}$$

This implies that we can integrate over space to get the total magnetic energy:

$$U_\text{B} = \frac{1}{2\mu_0}\iiint B^2 \, \text{d}V \tag{13}$$

Example 1.6. Using the set-up of Example 1.5, verify that the energy stored in the solenoid is consistent between Equations (11) and (13).

Using Equation (11),

$$U = \frac{1}{2}LI^2 = \frac{\mu_0 N^2 \pi r^2 I}{2l}$$

Using Equation (13), since the field is uniform inside the entire volume of the solenoid,

$$U = \frac{1}{2\mu_0}\iiint B^2 \, \text{d}V = \frac{B^2 V}{2\mu_0} = \frac{1}{2\mu_0}\left(\frac{\mu_0 N I}{l}\right)^2\left(\pi r^2 l\right) = \frac{\mu_0 N^2 \pi r^2 I}{2l}$$

Hence, the results for U given by both Equations (11) and (13) agree, as expected.

Remark. You may have some questions on how we approached the problem with Equation (13). In particular, you may be wondering:

1. What about the magnetic field outside the solenoid?
2. Why did we not consider that the magnetic field at the end of the solenoid is half of that at the centre?

Both of these can be resolved by knowing your assumptions – that the solenoid is long and ideal. Try to resolve them yourself!

1.2.3 Mutual Inductance

When two (or more) coils are involved, **mutual inductance** effects should be considered (unless otherwise stated). The premise for this is that changing the current in one coil will change the magnetic flux contributed by that coil passing through another coil.

Let Φ_{12} be the magnetic flux through coil 1 due to current through coil 2, and vice-versa for Φ_{21}. Then, we can define mutual inductances M_{12} and M_{21} as such:

$$M_{12} = \frac{\Phi_{12}}{I_1}, \; M_{21} = \frac{\Phi_{21}}{I_2} \tag{14}$$

Let the emf induced in coil 1 due to coil 2 be ε_{12}, and vice-versa for ε_{21}. They are given by

$$\varepsilon_{12} = -M_{12}\frac{\text{d}I_2}{\text{d}t}, \; \varepsilon_{21} = -M_{21}\frac{\text{d}I_1}{\text{d}t} \tag{15}$$

It turns out that due to the reciprocity theorem or the Neumann formula, we have

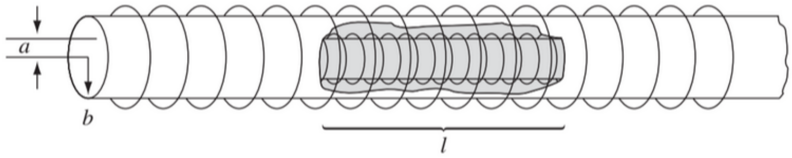
$$M = M_{12} = M_{21} \tag{16}$$

hence, the mutual inductance between a pair of coils does not depend on which coil is the one causing the change in flux!

Remark. You do not need to know how the proofs work; you just need to know that both mutual inductances are the same between two coils.

Let's see how to use mutual inductance with the example below.

Example 1.7 (Griffiths)**.** A **short** solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in the figure below. Current I flows in the short solenoid. What is the mutual inductance?



Since the inner solenoid is short, it's field is very complicated. It is not as simple as $B = \mu_0 n I$ due to this. As such, if you wanted to integrate over each current loop and find the total flux this way, you will be in for a lot of misery.

The easier way is to exploit the symmetry of mutual inductances. Look at the reverse situation: let current I flow in the long solenoid, and calculate the flux through the short one.

The field in the long solenoid is exactly what we know:

$$B = \mu_0 n_2 I$$

hence, the total flux through all the loops of the short solenoid is

$$\Phi_{\text{short}} = (n_2 l)\, B\pi a^2 = \mu_0 \pi a^2 n_1 n_2 I l$$

This is the same flux that a current I in the short solenoid would have put through the long one. Hence, the mutual inductance is

$$M = \frac{\Phi_{\text{short}}}{I} = \mu_0 \pi a^2 n_1 n_2 l$$

1.3 Ideas

Many tricky electromagnetism problems involve the use of the following ideas.

1.3.1 Superconductors

Superconductors are conductors that have been cooled down to extremely low temperatures. This presents a few implications for them:

1. They have zero resistivity. (Just like regular conductors too.)
2. All the existing magnetic flux in them is forced out, regardless of what the initial conditions are. If there is already a non-zero flux in a superconducting state, the flux will be conserved. This is known as the Meissner effect.
3. All the current in a superconductor must be confined to its surface.
4. The normal component of the magnetic field must vainsh on a superconductor's surface.

Let's illustrate these rules with a few examples.

Example 1.8 (200 More Puzzling Physics Problems)**.** Two identical superconducting rings are initially very far from each other. The current in the first is I_0, but there is no current in the other. The rings are now slowly brought closer together. Find the current in the first ring when the current in the second is I_1.

The flux through each ring must be conserved since they are superconductors. Let the mutual inductance be M and the self-inductances be L, and let the final current through the first ring be I_f. Then, conserving flux through both rings,

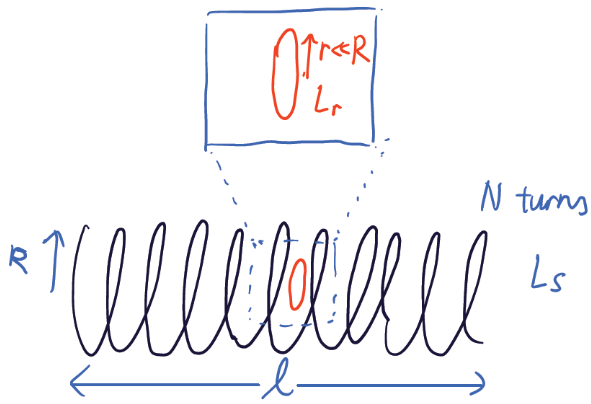
$$\text{Ring 1: } I_0 L = I_f L + I_1 M$$

$$\text{Ring 2: } 0 = I_f M + I_1 L$$

These simultaneous equations can be solved to obtain I_f. Rejecting the negative (unphysical) root, we eventually obtain

$$I_f = \frac{I_0 + \sqrt{I_0^2 + 4I_1^2}}{2}$$

Example 1.9 (Teo Kai Wen)**.** Consider a long solenoid with N turns, radius R and length l. The inductance of the solenoid is L_s. Then, a tiny superconducting ring of radius $r \ll R$ and inductance L_r is brought in from infinity to the centre of the solenoid, along its axis. As a result, the self-inductance of the solenoid changes by δL_s. (i) Find the mutual inductance of the set-up. (ii) Find the ratio $\frac{\delta L_\text{s}}{L_\text{s}}$.



(i) To calculate this, we have two options:

$$M = \frac{\Phi_\text{solenoid}}{I_\text{ring}} \quad \text{or} \quad M = \frac{\Phi_\text{ring}}{I_\text{solenoid}}$$

Clearly, the latter is much easier to compute, since finding the current in the superconductor will be extremely painful. Hence, we have

$$M = \frac{\Phi_\text{ring}}{I_\text{solenoid}} = \frac{B_\text{solenoid}\left(\pi r^2\right)}{I_\text{solenoid}} = \frac{\left(\frac{\mu_0 N I_\text{solenoid}}{l}\right)\left(\pi r^2\right)}{I_\text{solenoid}} = \frac{\mu_0 N \pi r^2}{l}$$

(ii) The magnetic flux through the superconducting ring must be conserved. Since it was initially at infinity, it initially had 0 flux, hence it must have 0 flux at the end too.

However, clearly, a solenoidal current will cause some flux to pass through the ring. This means that there must be some induced current in the ring to **oppose** this flux, so that the net flux is 0.

Suppose this induced current is i. Then,

$$\Phi_{\text{ring}} = L_{\text{r}} i$$

Also, by definition of mutual inductance,

$$M = \frac{\Phi_{\text{solenoid}}}{I_{\text{ring}}} \quad \Rightarrow \quad \Delta\Phi_{\text{solenoid}} = Mi$$

Hence, the new self-inductance is

$$L_{\text{s,new}} = \frac{\Phi_{\text{solenoid,new}}}{I_{\text{solenoid}}} = \frac{\Phi_{\text{solenoid}} - \Delta\Phi_{\text{solenoid}}}{I_{\text{solenoid}}}$$

Thus, the desired ratio, to first order, is

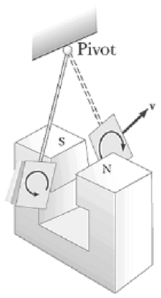
$$\frac{\delta L_{\text{s}}}{L_{\text{s}}} = \frac{L_{\text{s,new}} - L_{\text{s,old}}}{L_{\text{s,old}}} = \frac{\frac{\Phi_{\text{solenoid}} - \Delta\Phi_{\text{solenoid}}}{I_{\text{solenoid}}} - \frac{\Phi_{\text{solenoid}}}{I_{\text{solenoid}}}}{\frac{\Phi_{\text{solenoid}}}{I_{\text{solenoid}}}} = -\frac{\Delta\Phi_{\text{solenoid}}}{\Phi_{\text{solenoid}}}$$

$$= -\frac{Mi}{\Phi_{\text{solenoid}}} = -\frac{M\frac{\Phi_{\text{ring}}}{L_{\text{r}}}}{\Phi_{\text{solenoid}}} = -\frac{\left(\frac{\mu_0 N \pi r^2}{l}\right)\left(\frac{B_{\text{solenoid}}\left(\pi r^2\right)}{L_{\text{r}}}\right)}{B_{\text{solenoid}}\left(N\pi R^2\right)} = -\frac{\mu_0 \pi r^4}{L_{\text{r}} R^2 l}$$

1.3.2 Eddy Currents

Eddy currents are loops of induced current in conductors due to a changing magnetic flux (usually caused by a changing magnetic field). These currents are usually undesirable since power is dissipated, causing loss in efficiency through heating.

Consider a metal plate swinging in and out of a uniform magnetic field.



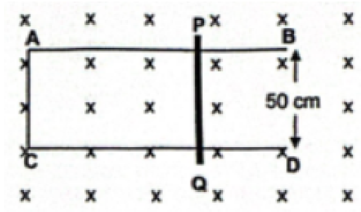
The induced current will then further interact with the external magnetic field, causing a **retarding force** on the plate. This damps the motion of the plate.

In many situations, eddy currents cannot be calculated analytically. However, in some very nice and symmetric situations, you can find the current loops.

2 Problems

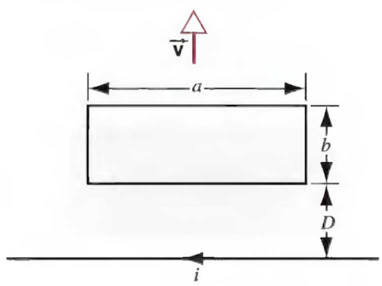
Problems are arranged in roughly increasing difficulty.

Problem 2.1. A conducting rod PQ of length 60cm makes contact with the metal rails AB and CD that are 50cm apart in a uniform magnetic field of flux density $B = 0.50$ T perpendicular to the plane of the paper, as shown below. The resistance of PQ is 0.30Ω and that of the rails is negligible.

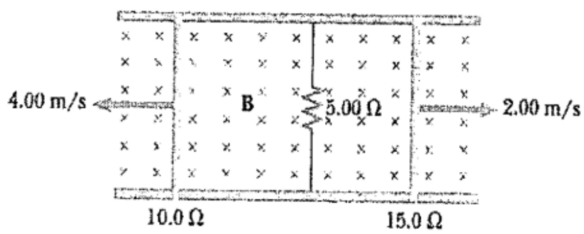


(a) Determine the magnitude of the current induced in the rod when it is moved to the right with a velocity of 5.0 m/s, and state its direction in the rod PQ.
(b) Determine the magnitude of the applied force required to keep the rod in motion.
(c) Compare the rate at which mechanical work is done by the applied force with the rate of thermal energy dissipated in the circuit.

Problem 2.2. A rectangular loop of wire with length a, width b, and resistance R is placed near an infinitely long wire carrying current i, as shown in the figure below. The distance from the long wire to the loop is D. Find
(a) the magnitude of the magnetic flux through the loop, and
(b) the current in the loop as it moves away from the long wire with speed v.

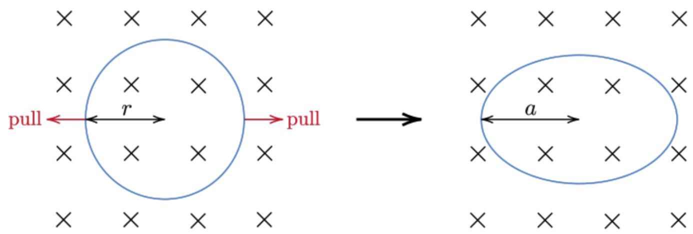


Problem 2.3 (SPhO 2013)**.** Two parallel rails with negligible resistance are 10.0 cm apart and are connected by a 5.00 Ω resistor. The circuit also contains two metal rods having resistances of 10.0 Ω and 15.0 Ω sliding along the rails (see the figure below). The rods are pulled away from the resistor at constant speeds of 4.00 m/s and 2.00 m/s, respectively. A uniform magnetic field of magnitude 0.0100 T is applied perpendicular to the plane of the rails. Determine the current in the 5.00 Ω resistor.

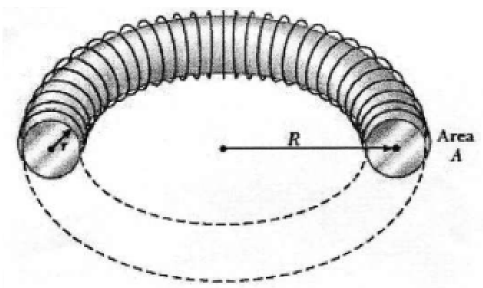


Problem 2.4 (SPhO 2013)**.** On a smooth and insulating large ring of radius R, there is a small ring of mass m and carrying charge q. The large ring is placed horizontally and in a uniform magnetic field of strength B_0 and perpendicular to the ring plane. Starting at $t = 0$, the magnetic field is changed to $B(t) = B_0 + \alpha t$. Find the force of the small ring acting on the big ring afterwards and describe the motion of the small ring.

Problem 2.5 (SPhL 2023)**.** A circular loop of wire with radius r and resistance R is placed in a uniform magnetic field B perpendicular to the plane of the loop. The wire is pulled at opposite ends outwards such that it now forms an ellipse with semi-major axis a. How much charge Q flows through the wire during this process? Approximate the perimeter p of an ellipse with semi-major axis a and semi-minor axis b as $p \approx 2\pi\sqrt{\frac{a^2+b^2}{2}}$.



Problem 2.6 (SPhO 2010)**.** A toroid has a major radius R and a minor radius r and it is tightly wound with N turns of wire, as shown in Figure 6. If $R \gg r$, the magnetic field in the region enclosed by the wire of the torus, of cross-sectional area $A = \pi r^2$, is essentially the same as the magnetic field of a solenoid that has been bent into a large circle of radius R.

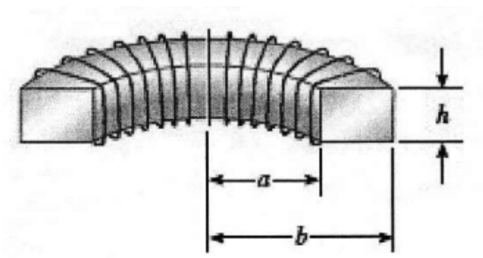


(a) Show that the self-inductance of such a toroid is approximately

$$L \approx \kappa\mu_0 \frac{N^\alpha A}{R}$$

where κ and α are constants. State the values of κ and α.

The original toroid is now replaced by one with a rectangular cross section. Its inner and outer radii are a and b, respectively. The cross-section is a rectangle of length $b - a$ and breadth h.

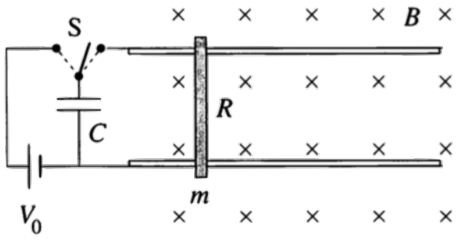


(b) Show that the inductance of the toroid is

$$L = \kappa' \mu_0 \frac{N^\beta h}{R} \ln \frac{b}{a}$$

where κ' and β are constants, stating the values of κ' and β.

Problem 2.7 (200 PPP)**.** One end of a conducting horizontal track is connected to a capacitor of capacitance C charged to voltage V_0. The inductance of the assembly is negligible. The system is placed in a uniform vertical magnetic field B, as shown.

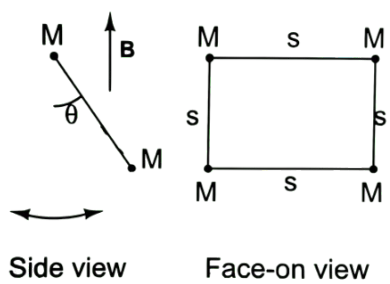


A frictionless conducting rod of mass m, length ℓ, and resistance R is placed perpendicularly onto the track. The capacitor is charged so that the rod is repelled from the capacitor when the switch is turned. This arrangement is known as a railgun. Neglect self-inductance throughout this problem.

(a) What is the maximum velocity of the rod, and what is the maximum possible efficiency?

(b) At the end of this process, the rail is moving to the right. Therefore, by momentum conservation, something must have experienced a force towards the left. What is it? Answer this in both the case where the magnetic field is the same everywhere, and when it only overlaps the rails, as shown above.

Problem 2.8 (Kevin Zhou)**.** A square, rigid loop of wire has resistance R, sides of length s, and negligible mass. Point masses of mass M are attached at each corner. The top edge of the square loop is mounted so it is horizontal, and the loop may rotate as a frictionless pendulum about a fixed axis passing through this edge. Initially the pendulum is at rest at $\theta = 0$, and a uniform magnetic field $\mathbf{B}$ points horizontally through the loop. The magnetic field is then quickly rotated to the vertical direction, as shown.

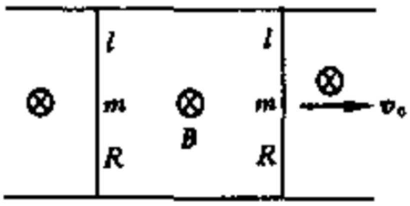


Describe the subsequent evolution.

Problem 2.9. Consider a rectangular loop of wire with dimensions a by b. The long side of the rectangle is parallel to another long straight wire carrying current I_o at a distance l away from the closest edge of the rectangle. The resistance of the loop is R, and you are to neglect self-inductance in this problem. Determine the net momentum p imparted to the loop when the current is switched off.

Problem 2.10. USAPhO 2023, Problem B1.

Problem 2.11. As shown in the figure, two metallic rods of mass m, length l and resistance R lie on a set of parallel tracks, which can be thought of as perfectly conducting. A uniform magnetic field B is directed into the page. Neglect friction, gravity and electromagnetic radiation. At time $t = 0$, the rod on the right moves towards the right with an initial velocity v_0. The left rod is free to move. Suppose the rods have an initial separation x_0.



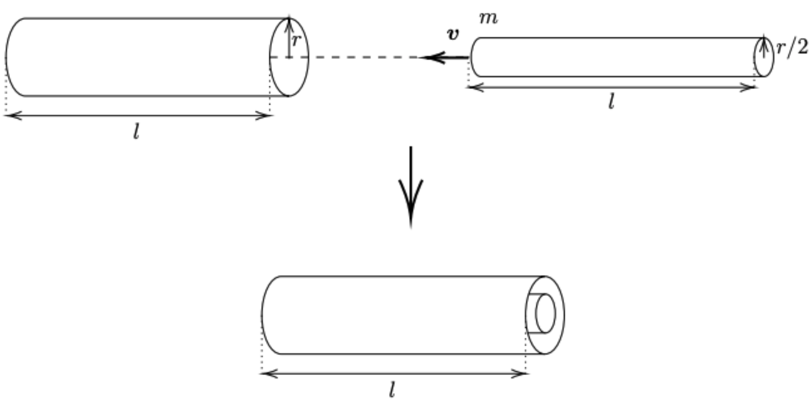
(a) Determine how the velocity $v(t)$ of the rod on the right.

(b) Determine the maximum horizontal separation x_{max} the two rods will achieve throughout their motion.

Problem 2.12 (SPhL 2024)**.** A fixed superconducting cylindrical shell A of radius $r = 0.05\,\text{m}$ and length $l = 100\,\text{m}$ initially has uniform current flowing in its azimuthal direction with total magnitude $I = 500\,\text{A}$. A second superconducting cylindrical shell B of mass $m = 2.5 \times 10^{-6}$ kg, radius $\frac{r}{2}$ and length l initially has no current flowing through it. It is positioned infinitely far away from shell A, and is allowed to move.

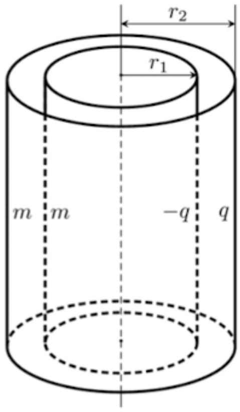
Shell B is launched towards shell A with initial velocity v. The axes of symmetry of the two cylinders are aligned throughout the subsequent motion of shell B. Determine the velocity v such that shell B will fill the length of the blank space within shell A exactly, after a long time.

Hint: The magnetic flux through a superconductor is conserved.



Problem 2.13. A circular ring of thin copper wire is set rotating rapidly about a vertical diameter at a point within the Earth's magnetic field. The Earth's magnetic field at this point is B directed at an angle of θ below the horizontal; the density of copper is ρ_d and its resistivity is ρ_c. Calculate how long it will take for the angular velocity of the ring to halve. You may assume that this time is much longer than the time for one revolution, that the frictional effects of the supports and air are negligible, and for the purposes of this question you may ignore self-inductance effects.

Problem 2.14. As shown in the figure, two infinitely long uniformly charged thin-walled cylinders of radii r_1 and r_2 are located coaxially in a vacuum. The mass and charge of the outer and inner cylinders per unit length are m and $\pm q$ respectively. The cylinders can rotate freely around the central axis.



There is a typo in the diagram, the outer cylinder should have charge $-q$ and inner cylinder $+q$.

(a) Find the electric field strength $E(r)$ at a distance r from the axis of the cylinders.

(b) Find the magnetic induction $B(r)$ in all space when the cylinders rotate in the same direction with the same angular velocity ω.

(c) Directly above the surface of the inner cylinder ($r = r_1$), a point charge of mass m and charge Q (the sign matches the sign of q) is released from a state of rest. At what ω may the charge reach the outer cylinder ($r = r_2$)?

Now, let the cylinders be initially stationary, and consider a moment applied to the outer cylinder. After some time, the angular velocity of the outer cylinder reaches Ω.

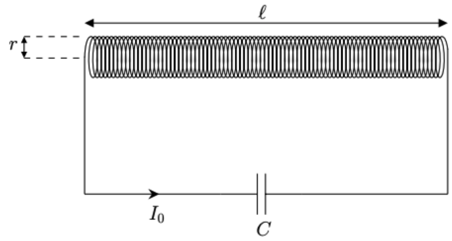
(d) Find the angular velocity of rotation of the inner cylinder.

(e) Find the total angular momentum J per unit length, transmitted to the second cylinder during spinning by an external moment (excluding the moment of force from the induced electromagnetic field).

(f) Find the total mechanical angular momentum L of both cylinders per unit length.

3 Advanced Problems

Problem 3.1 (SPhL 2024). In the following circuit, a very long solenoidal inductor with $n_0 = 5000m^{-1}$ turns per unit length is connected to a capacitor in series. The length ℓ and radius r of the inductor satisfy $\ell \gg r$, so only the magnetic field inside the inductor needs to be considered. The initial maximum current in the circuit is $I_0 = 1.00\,\text{A}$, and the initial period of oscillations of the current is T_0.

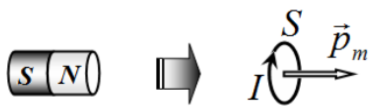


At time $t = 0$, there is no current flowing through the wires. At this instant, the inductor is pulled on and stretched out, such that the turns per unit length decreases to $n_1 = 2000m^{-1}$. This process is done in time t_p. Find the new maximum electric current I_1 in the circuit if:

(a) $t_p \ll T_0$.
(b) $t_p \gg T_0$.

You may assume the turns remain equally spaced apart throughout this process. Leave your answer to 3 significant figures in units of A.

Problem 3.2 (IZhO 2015, modified). The magnetic field generated by a uniformly magnetised ferromagnetic cylinder (a permanent magnet) is equivalent, at very large distances, to the field produced by a circular coil with constant electric current. The cylindrical magnet, as well as the coil with the current, are characterised by the magnetic moment p_{m}, which is defined for the current loop $p_{\text{m}} = IS$, where I is the current and S is the area. This source of magnetic field is known as a magnetic dipole.

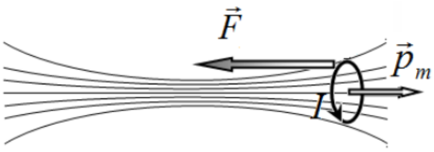


(i) Show that the magnetic field, B_{z}, on the axis of the dipole is determined at large distances $z \gg \sqrt{S}$ from the centre is given by the formula

$$B_{\text{z}} = b\frac{p_{\text{m}}}{z^{\beta}}$$

where b and β are constants to be determined.

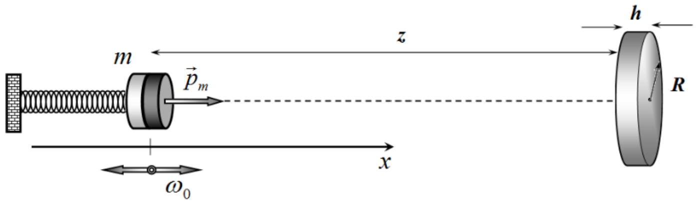
(ii) Let the coil with current such that it has magnetic moment p_{m} be influenced by an **inhomogeneous axially symmetric** field, in which the magnitude along the z-axis only depends on the value of z.



Show that the force acting on the dipole is given by

$$F_{\mathrm{z}} = -p_{\mathrm{m}} \frac{\mathrm{d}B_{\mathrm{z}}}{\mathrm{d}z}$$

(iii) A cylindrical magnet of mass m and magnetic moment p_{m} is attached to a spring of spring constant k, such that it can oscillate along the horizontal axis which is directed along the magnetic moment. At some distance z from the equilibrium position of the magnet, a small metal disk is placed such that its axis coincides with the axis of the magnet. The disk has radius R and thickness h ($h \ll R \ll z$), and the electrical resistivity of the disk is ρ. Assume the set-up is placed in a vacuum, so that $\mu = \mu_0$. The magnet is moved from the equilibrium position and starts performing small oscillations from equilibrium, described by $x(t)$, where $x \ll z$. Find the force $F(x, v)$ exerted by the disk on the magnet as a function of its coordinate x and velocity v. Hence write down the equation of motion for the magnet, and find the angular frequency of small oscillations. Ignore gravity.



(iv) Verify that the loss in mechanical energy of the magnet is equal to the amount of heat released in the disk for the same time period.